The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla \Phi$ and $\mathbf{u} = \nabla \phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   A plate lies along the positive x-axis. There is an exterior flow past the plate given by

   $$U(x) = Cx.$$

   By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

   $$\psi = Ax^\alpha f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^\beta},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + ff'' - (f')^2 + 1 = 0,$$

   where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

2. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder. Show from this complex potential that the flow does have a streamline around the cylinder.

   Show how the Milne-Thompson theorem can be used to derive the complex potential for the uniform flow past a circle of radius a centred at the origin with the flow in the far field parallel to the real axis.

   If circulation Γ around the cylinder is added the complex potential becomes

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z.$$

   Find the value of the circulation, Γ, such that there is only one stagnation point of the flow on the cylinder.

   Use Blasius's theorem

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z$$

   to find the force exerted on the cylinder for a general circulation.

3. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x,t) = \epsilon\cos(kx - \omega t)\,.$$

   Find the potential of the flow assuming the fluid is of uniform depth h in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

   Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk\tanh kh,$$

   where g is the acceleration due to gravity.

   Find the group velocity of these waves. Show that for waves with wavelength short compared to the depth the group velocity is approximately half the phase velocity, but for long waves both the group and phase velocities are approximately the same.

Turn over ...

4. A viscous fluid flows along the inside of a long circular cylinder of radius a whose axis lies along the z-axis. What are the boundary conditions for the fluid at the point of contact with the cylinder? Show that, subject to certain assumptions that are to be stated, the z-component of the velocity satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = \frac{G}{\rho\nu}$$

where G is the imposed pressure gradient in the z direction.

If the cylinder was lifted so that its axis were vertical, and the flow was driven by gravity and not by any imposed pressure gradient, how would the right hand side of the above equation be altered?

Find the flow under the effect of gravity in the cylinder and calculate the volume of water that would flow down the pipe every second.

A down-pipe is used to drain water from a roof. The pipe is expected to cope with heavy rain falling at a rate rate of 72 mm in an hour onto a flat roof of area $10\,\mathrm{m}^2$. Calculate the minimum radius that the cylinder must have in order that the water would drain down the pipe according to your solution. You may assume $\rho = 10^3\,\mathrm{kg\,m^{-3}}$, $g = 10\,\mathrm{m\,s^{-2}}$ and $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$. Comment on the validity of this solution.

[Hint: $\nabla^2(a^2 - x^2 - y^2) = -4$]

5. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

A steady flow is given by

$$\mathbf{u}(x,y,z) = \left(\frac{axz}{b^2}, \frac{ayz}{b^2}, a - \frac{a(2x^2+2y^2+z^2)}{b^2}\right),$$

where a and b are constant. Show that this flow in incompressible. Show also that on the sperical surface $x^2 + y^2 + z^2 = b^2$ the component of the velocity away from the origin vanishes.

Find the vorticity of the flow, and show it satisfies the vorticity equation.

[You may quote the result $\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\cdot\nabla\mathbf{u}$.]

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