The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla \Phi$ and $\mathbf{u} = \nabla \phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   A wedge points into an oncoming stream of fluid. With one of the faces of the wedge aligned along the positive x-axis, there is an exterior flow past the face given by

   $$U(x) = Cx^{1/3}.$$

   By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

   $$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + 2ff'' - (f')^2 + 1 = 0,$$

   where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

2. The complex potential for the uniform flow past a circle of radius a centred at the origin and with circulation Γ is

$$w(z) = U\left(e^{-i\alpha}z + \frac{a^2e^{+i\alpha}}{z}\right) - \frac{i\Gamma}{2\pi}\ln z,$$

where the far-field flow is inclined at an angle α to the x-axis. Verify that the mapping

$$\zeta = z + \frac{a^2}{z}$$

maps the circle $z = ae^{i\theta}$ in the z-plane onto the straight line between $\zeta = -2a$ and $\zeta = 2a$ in the ζ-plane. Hence write down the potential for the flow past a flat plate with circulation in the ζ-plane.

Find the value of the circulation, Γ, such that the singularity in the velocity at the trailing edge $\zeta = 2a$ is removed. What would the corresponding lift on the flat plate be?

3. An incompressible viscous fluid flows down a flat slope under the force of gravity. What are the boundary conditions for the fluid at the point of contact with the slope and at the free surface?

Using orthogonal coordinates with the x-axis pointing down the slope and the y-axis perpendicular to the slope, find a solution to the Navier-Stokes equation for a flow of depth d down the slope under the assumptions that the flow is steady and uniform in the x-direction.

If the flow rate was doubled, find the effect on the depth of the fluid, assuming that the flow remained uniform.

If, instead of having a free surface at $y = d$, there was another solid boundary, what would the ratio of the new flow rate to the original flow rate be? You may assume that there is still no pressure gradient in the y direction.

Turn over . . .

4. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x,t) = \epsilon \cos(kx - \omega t)\,.$$

Find the potential of the flow assuming the fluid is infinitely deep in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk,$$

where g is the acceleration due to gravity.

If surface tension is taken into consideration then the pressure of the fluid at the surface becomes

$$p = p_0 - T\frac{\partial^2 \eta}{\partial x^2}$$

where T is a positive constant. Including the effect of gravity, show that the relation between the frequency and the wavenumber becomes

$$\omega^2 = gk + \frac{k^3 T}{\rho}.$$

Determine the group velocity of these waves, and show that for very short waves (large k) and very long waves (small k) the group velocity increases without bound. Deduce that the group velocity has a minimum value and find the corresponding wavenumber k.

Turn over ...

5. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

A flow is given by

$$\mathbf{u}(x,y,z,t) = \left(u'(x,y,t),\ -Ay + v'(x,y,t),\ Az\right),$$

where A is constant. Show that the vorticity of this flow is of the form $\boldsymbol{\omega} = (0,\ 0,\ \omega_3(x,y,t))$ where ω_3 satisfies

$$\frac{\partial \omega_3}{\partial t} + u'\frac{\partial \omega_3}{\partial x} - Ay\frac{\partial \omega_3}{\partial y} + v'\frac{\partial \omega_3}{\partial y} = A\omega_3 + \nu\left(\frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2}\right).$$

Assume that the vorticity component ω_3 and the velocities u' and v' are small and so products of these quantities can be ignored. Show solutions exist where the vorticity takes the form

$$\omega_3 = \epsilon f(y) e^{\lambda t}\cos kx,$$

with ϵ, λ and k constant, finding the ordinary differential equation that $f(y)$ must satisfy.

For the special case $\lambda + \nu k^2 = A$ show that the general solution to this differential equation is

$$f(y) = B\int e^{-(Ay^2/2\nu)}\,dy + C,$$

where B and C are constants.

[You may quote the result $\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\cdot\nabla\mathbf{u}$.]

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