The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What are the appropriate boundary conditions to be applied to the flow in the boundary layer?

   A wedge points into an oncoming stream of fluid. With one of the faces of the wedge aligned along the positive x-axis, there is an exterior flow past the face given by

   $$U(x) = Cx^{1/4}.$$

   By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

   $$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + ff'' + \frac{2}{5}\left(1 - (f')^2\right) = 0,$$

   where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over . . .

2. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder. Show from this complex potential that the flow does have a streamline around the cylinder.

   A stagnation point flow with centre at c on the real axis has complex potential
$$w(z) = A(z-c)^2,$$
   where A and c are real and positive constants. A circular cylinder of radius a is introduced with its axis at $z=0$. Find the corresponding complex potential.

   Show that the points in the flow with zero velocity satisfy the quartic equation
$$z^4 - cz^3 + ca^2 z - a^4 = 0,$$
   and hence find the position of the four stagnation points (Hint: there are stagnation points at $z = \pm a$). Find the critical value of c such that below this value there are four such stagnation points on the surface of the cylinder, but above this value there are only two. Sketch the flow in each of these two cases.

3. A viscous fluid lies above a solid boundary located at $z=0$. Its velocity is given by
$$\mathbf{u}(x,y,z) = \left(Axz, Ayz, Bz^2 + Cz + D\right).$$
   Determine B, C and D in terms of A so that this flow is both incompressible and satisfies the appropriate boundary condition at the plane $z=0$. By considering its three components, show that this flow is an exact solution of the steady Navier-Stokes equation for some pressure, $p(x,y,z)$, which is to be determined.

   Determine the vorticity of the flow, showing that vorticity is everywhere parallel to the $z=0$ plane and perpendicular to a radial vector from the z-axis, and hence vortex lines form circles centred on the z-axis.

   Show that the ratio of the length of the vortex lines to the strength of the vorticity is constant.

Turn over ...

4. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x, t) = \epsilon \cos(kx - \omega t) .$$

Find the potential of the flow assuming the fluid is of uniform depth h in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk \tanh kh,$$

where g is the acceleration due to gravity.

If surface tension is taken into consideration, the pressure of the fluid at the surface becomes

$$p = P_0 - T \frac{\partial^2 \eta}{\partial x^2}$$

where T is a positive constant. Show that the relation between the frequency and the wavenumber is then

$$\omega^2 = \left(g + \frac{k^2 T}{\rho} \right) k \tanh kh.$$

Show that for $T = 0$ the ratio of the group velocity of the waves to the phase velocity tends to $1/2$ as the waves become shorter ($k \to \infty$), while for $T \neq 0$ it tends to $3/2$ as the waves become shorter.

Turn over . . .

5. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

An incompressible flow is given by

$$\mathbf{u}(x, y, z, t) = \left(-Ax + u'(x, y, t),\ -Ay + v'(x, y, t),\ 2Az\right),$$

where A is a positive constant and $(u', v', 0)$ is a perturbation velocity. Show that the vorticity of this flow is of the form $\boldsymbol{\omega} = (0,\ 0,\ \omega_3(x, y, t))$, where ω_3 satisfies

$$\frac{\partial \omega_3}{\partial t} - Ax\frac{\partial \omega_3}{\partial x} - Ay\frac{\partial \omega_3}{\partial y} + u'\frac{\partial \omega_3}{\partial x} + v'\frac{\partial \omega_3}{\partial y} = 2A\omega_3 + \nu\left(\frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2}\right).$$

(i) Show that if the vorticity is independent of position, i.e. $\omega_3 = f(t)$ for some function $f(t)$, then the vorticity grows exponentially.

(ii) Consider the case of a steady axi-symmetric flow around the z-axis. In this situation we have $xu' + yv' = 0$ and the vorticity is independent of t and depends only on the distance, $r = (x^2 + y^2)^{1/2}$, from the z-axis, i.e. $\omega_3 = g(r)$ with $x = r\cos\theta$ and $y = r\sin\theta$.
Show that $g(r)$ satisfies

$$Arg' + 2Ag + \nu\left(g'' + g'/r\right) = 0.$$

Hence show that

$$Ar^2 g + \nu r\frac{\mathrm{d}g}{\mathrm{d}r} = B,$$

where B is a constant. Why must B be zero?
Find the general solution to this differential equation with $B = 0$.

[You may quote the result $\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u}$.]

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