The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$

where the kinematic viscosity ν is related to the dynamic viscosity μ by $\mu = \nu\rho$ where ρ is the density. Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla \Phi$ and $\mathbf{u} = \nabla \phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What are the appropriate boundary conditions to be applied to the flow in the boundary layer?

   A broad wedge points into an oncoming stream of fluid. With one of the faces of the wedge aligned along the positive x-axis, there is an exterior flow past the face given by

   $$U(x) = Cx^{2/3}.$$

   By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

   $$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + ff'' + \frac{4}{5}\left(1 - (f')^2\right) = 0,$$

   where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

2. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder. Show from this complex potential that the flow does have a streamline around the cylinder.

   The complex potential for a vortex with circulation Γ situated at the origin is given by

$$w(z) = -\frac{i\Gamma}{2\pi}\ln z.$$

   Write down the circulation for a vortex with centre at $z = 2$.

   Use the Milne-Thompson theorem to find the potential for a vortex of circulation Γ centred at $z = 2$ in the presence of a cylinder of radius 1 centred at the origin.

   How fast, and in what direction, will the vortex at $z = 2$ move?

3. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x,t) = \epsilon\cos(kx - \omega t).$$

   Find the potential of the flow assuming the fluid is infinitely deep in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

   Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk,$$

   where g is the acceleration due to gravity.

   By considering two waves of the same amplitude and with slightly different wavenumbers and frequencies, show that the energy of the waves moves with the group velocity, which is given by

$$c_g = \frac{d\omega}{dk}$$

   Hence find the ratio of the phase velocity to the group velocity of waves on deep water.

Turn over ...

4. An incompressible viscous fluid flows down a flat slope of angle θ to the horizontal under the force of gravity, with g the acceleration due to gravity. What are the boundary conditions for the fluid at the point of contact with the slope and at the free surface?

   Using orthogonal coordinates with the x-axis pointing down the slope and the y-axis perpendicular to the slope, find a solution to the Navier-Stokes equation for a flow of depth d down the slope under the assumptions that the flow is steady and uniform in the x-direction, including an expression for the pressure.

   A river descends by 100m over a distance of 100km. Given that the dynamic viscosity of water is approximately $\mu = 10^{-3}\text{kg m}^{-1}\text{s}^{-1}$, estimate the predicted speed of the river using your own estimates for any other parameters involved.

   Comment on the realism of your answer. If it is unrealistic, give possible reasons for the lack of realism.

5. The velocity of a viscous fluid is given by

$$\mathbf{u}(x, y, z) = \left(Bx, By, -A(x^2 + y^2) + Cz\right).$$

   Determine C in terms of A and B to ensure that this flow is also incompressible.

   By considering its three components, show that this flow is an exact solution of the steady Navier-Stokes equation for some pressure which is to be determined.

   Determine the vorticity of the flow, showing that vortex lines are circles parallel to the $z = 0$ plane and with centre on the z-axis.

   Consider the flow in the plane $y = 0$. Write down a differential equation for the streamlines, and show that it is satisfied by

$$z = -Ax^2/(4B) + \alpha/x^2,$$

   with α an arbitrary constant. Sketch the streamlines of the flow in the plane $y = 0$.

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