The Navier-Stokes equation for an incompressible Newtonian fluid in the absence of body forces is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}.$$

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1. The complex potential for the uniform flow past a circle of radius a centred at the origin and with circulation Γ is

$$w(z) = U \left(e^{-i\alpha} z + \frac{a^2 e^{+i\alpha}}{z} \right) - \frac{i\Gamma}{2\pi} \ln z,$$

   where the far-field flow is inclined at an angle α to the x-axis. Verify that the mapping

$$\zeta = z + \frac{a^2}{z}$$

   maps the circle $z = ae^{i\theta}$ in the z-plane onto the straight line between $\zeta = -2a$ and $\zeta = 2a$ in the ζ-plane. Hence write down the potential for the flow past a flat plate with circulation in the ζ-plane.

   Find the value of the circulation, Γ, such that the singularity in the velocity at the trailing edge $\zeta = 2a$ is removed. What would the corresponding lift on the flat plate be?

2. The flow given by

$$\mathbf{u}(x, y, z, t) = (f(z, t), Ay, -Az)$$

   where A is a constant is an incompressible flow which satisfies the Navier-Stokes equation. Show that $f(z, t)$ satisfies

$$\frac{\partial f}{\partial t} - Az \frac{\partial f}{\partial z} = \nu \frac{\partial^2 f}{\partial z^2}. \qquad (1)$$

   Find the general time-independent solution with the pressure p a function of y and z only.

Show that

$$f(z,t) = \int_0^z \frac{U_0}{\sqrt{2}\delta} e^{-\zeta^2/(2\delta^2)} d\zeta$$

is a solution to the time-dependent equation where

$$\delta^2 = \left(\delta_0^2 - \frac{\nu}{A}\right) e^{-2At} + \frac{\nu}{A},$$

where $\delta = \delta_0$ at $t = 0$.

[You may find the last part easier if you first differentiate (1) partially with respect to z and show that $\frac{\partial f}{\partial z}$ satisfies the resulting expression.]

Turn over ...

3. State the Milne-Thompson theorem for the complex potential of a flow past a circle. Verify that the edge of the circle is a streamline of the flow.

   The complex potential for a stagnation point flow centred on the origin is

$$w(z) = Az^2,$$

   where A is a constant. Find the corresponding potential if a cylinder of radius a is placed at the origin.

   If the circulation due to a vortex at the origin with potential

$$-\frac{i\Gamma}{2\pi}\ln z$$

   is superimposed, find the circulation, Γ, for which there are exactly two stagnation points on the surface of the cylinder.

   [You may quote the result that the gradient of a function ϕ in polar coordinates (r, θ) is

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}.$$

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4. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   The flow outside a boundary layer is given by $U = ae^{bx}$ where a and b are constants. By seeking a similarity solution of the form

$$\psi = \lambda(x)F(\eta), \qquad \eta = y\theta(x),$$

   where ψ is the stream function, show that the governing equation for the flow in the boundary layer may be reduced to the ordinary differential equation

$$F''' + FF'' + 2(1 - F'^2) = 0,$$

providing the functions $\lambda(x)$ and $\theta(x)$, which are to be determined, are chosen appropriately. State the boundary conditions satisfied by $F(\eta)$.

Turn over ...

5. An incompressible fluid lies between two fixed plates at $y = 0$ and $y = h$ and is subject to a uniform unsteady pressure gradient in the x-direction

$$\frac{\partial p}{\partial x} = \mathcal{R}\left\{u_0 \omega \rho e^{i\omega t}\right\},$$

where $\mathcal{R}$ denotes the real part, u_0 and ω are constants and the density of the fluid is ρ. Show that there is a solution of the Navier-Stokes equations in which

$$u = \mathcal{R}\left\{\overline{u}(y) e^{i\omega t}\right\}, \quad v = w = 0$$

where u, v and w are the velocity components in Cartesian coordinates (x, y, z) and the fluid satisfies the no-slip conditions at the walls.

Show that

$$\overline{u}(y) = i u_0 \left(1 - \frac{e^{\alpha\eta}}{1 + e^{\alpha}} - \frac{e^{-\alpha\eta}}{1 + e^{-\alpha}}\right),$$

where

$$\alpha = (1 + i) A^{1/2}, \quad \eta = y/h, \quad A = \frac{h^2 \omega}{2\nu}$$

where ν is the kinematic velocity.

Show that for large A the velocity profile is approximately uniform except for a region near the wall whose width is of order $A^{-1/2}$. Show that the oscillations in this uniform core flow lag behind the oscillation in the pressure gradient by a quarter of the phase.

Internal Examiner: Dr O.S. Kerr
External Examiners: Professor J.H. Merkin
Professor S.A. Robertson