The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\mathbf{u}\right|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Waves can travel on the surface of a fluid due to the effects of surface tension even in the absence of gravitational effects.

   The surface of an inviscid irrotational fluid is given by

   $$y = \eta(x,t) = \epsilon\cos\left(kx - \omega t\right).$$

   Find the potential of the flow assuming the fluid is infinitely deep in the negative y-direction and that ϵ is small and so terms proportional to ϵ^2 and smaller can be ignored.

   If the effect of gravity can be ignored, the pressure of the fluid at the surface is given by

   $$p = P_0 - T\frac{\partial^2\eta}{\partial^2 x}$$

   where T is a positive constant. Find a relation between the frequency ω and the wave number k.

   Determine the relative magnitudes of the group velocity and the phase velocity of such surface tension waves.

Turn over ...

2. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

A flow is given by

$$\mathbf{u}(x, y, z, t) = (u'(x, y, t),\ -Ay + v'(x, y, t),\ Az),$$

where A is constant. Show that the vorticity of this flow is of the form $\boldsymbol{\omega} = (0,\ 0,\ \omega_3(x, y, t))$ where ω_3 satisfies

$$\frac{\partial \omega_3}{\partial t} + u' \frac{\partial \omega_3}{\partial x} - Ay \frac{\partial \omega_3}{\partial y} + v' \frac{\partial \omega_3}{\partial y} = A\omega_3 + \nu \left(\frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2} \right).$$

Assume that the vorticity component ω_3 and the velocities u' and v' are small and so products of these quantities can be ignored. Show solutions exist where the vorticity takes the form

$$\omega_3 = \epsilon f(y) e^{\lambda t} \cos kx,$$

with ϵ, λ and k constant, finding the ordinary differential equation that $f(y)$ must satisfy.

For the special case $\lambda + \nu k^2 = A$ show that the general solution to this differential equation is

$$f(y) = B \int e^{-(Ay^2/2\nu)}\, dy + C,$$

where B and C are constants.

[You may quote the result $\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u}$.]

Turn over ...

3. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder.

   Find the stream function for the flow with complex potential

$$w(z) = A(z - c)^2,$$

   where A and c are real and positive constants. Hence or otherwise sketch the flow.

   A circular cylinder of radius a is introduced with its axis at $z = 0$. Find the corresponding complex potential.

   Show that the points in the flow with zero velocity satisfy the quartic equation

$$z^4 - cz^3 + ca^2 z - a^4 = 0,$$

   and hence find the position of the four stagnation points (Hint: there are stagnation points at $z = \pm a$). Find the critical value of c such that below this value these are four such stagnation points on the surface of the cylinder, but above this value there are only two. Sketch the flow in each of these two cases.

4. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   A plate lies along the positive x-axis. There is an exterior flow past the plate given by

$$U(x) = Cx^m.$$

   By looking for a solution to the boundary layer equations in terms of the stream function ψ which takes the form

$$\psi = Ax^\alpha f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^\beta},$$

   show that the problem can be reduced to the ordinary differential equation

$$f''' + ff'' + \frac{2m}{m+1}\left(1 - f'^2\right) = 0,$$

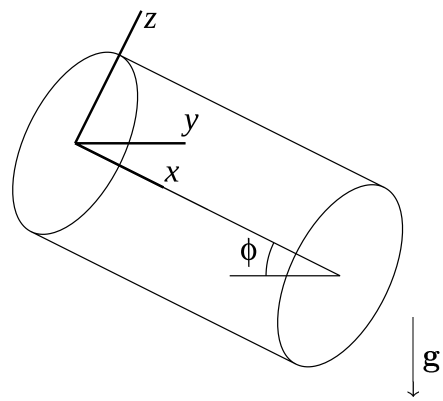
   where A, B, α and β are constants that are to be determined.

   What are the boundary conditions satisfied by $f(\eta)$?

   Confirm that when the exterior flow is a stagnation point flow with $m = 1$ the boundary layer is of constant thickness.

Turn over ...

5. An incompressible viscous fluid flows down a long circular tube of radius a inclined at an angle ϕ to the horizontal. The absolute pressures at the ends of the tube are the same.



Using orthogonal coordinates with the x-axis aligned with the axis of the tube and the y-axis horizontal as indicated, find a solution for the flow along the tube under the assumptions that the flow is steady and uniform along the tube and the fluid fills the tube. Find the total flow rate of fluid out of the pipe.

Calculate the pressure in the tube, verifying that surfaces of constant pressure are of the form $z = \text{constant}$.

Another flow that can be derived from the above solution is the uniform flow down a tube half filled with fluid with the rest filled with air. What are the appropriate boundary conditions for the fluid at its surface? Why can this solution *not* be used to find flows down the tube which are exact solutions of the Navier-Stokes equation when the fluid fills less than half the tube?

[Hint: $\nabla^2(a^2 - y^2 - z^2) = -4$]

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