Chip removal. Urban Renewal revised

V. Aksenov, K. Kokhas

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Abstract

Chip removal. Urban Renewal revised. V. Aksenov, K. Kokhas.

We describe a new combinatorial-algebraic transformation on graphs that we call «chip removal». It generalizes the well known Urban Renewal trick of Propp and Kuperberg. Chip removal is useful in calculations of determinants of adjacency matrices and matching numbers of graphs. A beautiful example of this technique is the theorem about removing four-contact chips, this theorem generalizes Kuo's graphical condensation method. Numerous examples are given.

Key word!!!: determinant of adjacency matrix, matching number, «Urban renewal», pfaffian, combinatorial linear algebra

1 *Introduction*

Let G be an arbitrary connected (undirected) graph, and let $A(G)$ be it's adjacency matrix. Graph G could contain loops and could be weights, in this case $A(G)$ is an arbitrary symmetrical matrix.

In this paper we introduce a combinatorial technique for calculating determinants $\det A(G)$, that generalizes «Urban Renewal» trick created by Kuperberg and Propp, used for counting the number of matchings of the graph. These techniques are both special cases of diagonalization of block matrices, and they have a very transparent combinatorial background.

To calculate the determinant $\det A(G)$ we introduce a special operation – *chip-removal*. We assume that *chip* H is an arbitrary induced subgraph of the graph G. We call a vertex of the chip, which has an outgoing edge (i.e. its second endpoint does not belong to H) *external* the second endpoint of an external edge is a *contact*. The operation chip-removal consists od two steps: 1) we remove the chip H and all its external edges from the graph, and 2) after that we "repair" the remaining part of the graph by joining some contacts with a new weighted edges. The location and weights of the new edges depend on the chip.

Denote by G' the graph obtained by the chip-removal. The main property of the chip-removal operation is

$$\det A(G) = \det A(H) \det A(G').$$

In the second section we describe a modification of Urban Renewal for calculating determinants. In the third section we describe a general scheme of chip-removal, and in the fourth section we explore the details of rectangle-chip removal. All these constructions are accompanied by examples. In the fifth section we apply this technique for calculating of the number of matchings and in particular we prove the theorem about 4-contacted chip-removal, which generalizes Kuo's graphical condensation.

2 Urban Renewal for determinants

Let G be an (undirected) graph, the weights of its edges are arbitrary real numbers. *A matching* is a set of the graph edges, which splits the whole set of its vertices into pairs, *a weight of the matching* is a product of weights of its edges. Let $M(G)$ be the sum of weights of all matchings of the graph G.

Lemma. *Let graph G contains a subgraph H, depicted in the picture 1 on the left, the graph could contain also edges, which are not showed in the picture, but all the black vertices have degree 3, i.e. all their edges are showed. Let us replace this subgraph by the new subgraph H', depicted in the picture 1 on the right, where the new weights are given by the formulae $x' = \frac{y}{xy+wz}$, $y' = \frac{x}{xy+wz}$, $z' = \frac{w}{xy+wz}$ and $w' = \frac{z}{xy+wz}$. We denote the resulting graph by G'. Then*

$$M(G) = (xy + zw)M(G').$$

This statement is proved by Kuperberg and Propp [5], they call this operation Urban renewal.

We may assume, that the graph G is directed, interpreting each undirected edge as a pair of edges with opposite directions. *1-factor* of the graph G is a directed subgraph, which contains all the vertices of G, and such that each vertex has in-degree 1 and out-degree 1. The combinatorial definition of $\det A(G)$ is very well known (see [1]), it can be calculated by the formula

$$\det A(G) = \sum_{\pi} (-1)^{\sigma(\pi)} W(\pi),$$

where the sum runs over the set of all 1-factors of the graph G, $W(\pi)$ is the weight of 1-factor π, and $\sigma(\pi)$ is the number of even cycles in π.

Lemma 2.1. *Let graph G contain subgraph H, depicted in the picture 1 on the left; the graph could countain also edges, which are not shown in the picture, but all the black vertices have degree 3, i.e. all*

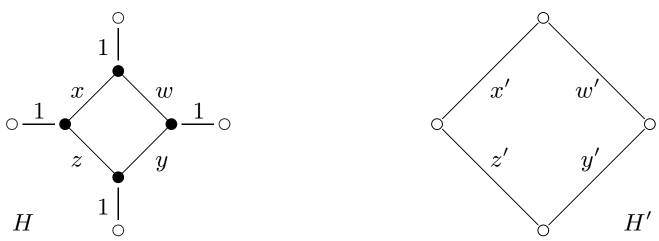


Figure 1. Urban renewal

Figure 2. Rebuilding of 1-factors

their edges are shown. Let us replace this subgraph by the new subgraph H', depicted on the picture 1 on the right, where the new weights are given by the formulas

$$x' = \frac{y}{wz - xy}, \quad y' = \frac{x}{wz - xy}, \quad z' = \frac{w}{xy - wz}, \quad w' = \frac{z}{xy - wz}. \tag{1}$$

Then

$$\det A(G) = (xy - wz)^2 \det A(G'). \tag{2}$$

Proof. Split all the 1-factors of graphs G (and G') into groups, such that in every group the intersection of factors with subgraphs H (and H' respectively) are the same. We will construct a bijection between the groups (and sometimes between individual 1-facors), which preserves the total weight.

1) If 1-factor of the graph G has a cycles that pass through edges x and y of the subgraph H, we map this 1-factor to the 1-factor of the graph G', in which parts of these cycles are replaced by the new edges x', y' (as in picture 2 on the left). The parts of the initial cycles in the subgraph H contribute xy to the weight of 1-factor. After the replacement the contribution is equal $x'y' = \frac{xy}{(wz-xy)^2}$, but we also have the multiplier $(wz - xy)^2$ on the right hand side of the equation (2). The total weight remains unchanged.

The case when the cycles contain edges w and z we consider analogously.

2) Each 1-factor of the graph G , which contains a long cycle, passing through edges x, w in the subgraph H, and a cycle of the length 2 on the vertical edge (see picture 2 on the right) we map to the 1-factor of the graph G' obtained by removing the 2-cycle and replacing the part of the long cycle by the new pair of edges z', y'. The weight preservation is checked like in the previous point.

We do analogously for the similar configurations.

3) Collect together all 1-factors of the graph G, which coincide outside the subgraph H and contain a cycle, passing through edges z, y, w, (its contribution of this configuration to the weight of 1-factor equals yzw) OR contain a long cycle, that passes through x and 2-cycle that occupies the edge y (this configuration contributes xy^2 to the total weight). Observe that these two configurations have opposite signs, because the numbers of cycles in them differ by 1. We map this set of 1-factors to the set of 1-factors in G, which have the same structure outside the subgraph H (and so the contribution of the outer part is the same in both sets of 1-factors), and contain the edge x' (picture 3). So again we have the equality of the weights, because $yzw - xy^2 = (wz - xy)^2 x'$.

4) We consider the remaining cases analogously (picture 4). Let us note, that the right cycle, depicted in the top of the picture 4, and the left cycle in the bottom of the picture 4 should be considered with two possible orientation, and that doubles their contribution. The equality of the weights on the picture 4 is due to the following identities $1 = (wz - xy)^2 (x'^2 y'^2 + w'^2 z'^2 - 2x'y'w'z')$ and $x^2y^2 + w^2z^2 - 2xywz = (wz - xy)^2$. $\square$

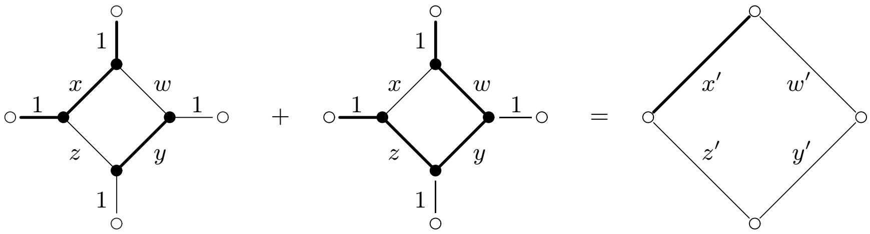


Figure 3. Rebilding the groups of 1-factors

Remark. The transformation $(x, y, z, w) \mapsto (x', y', z', w')$, given by the formula (1), is an involution. The multipliers from the formula (2) satisfy the condition $(xy - wz)^2(x'y' - w'z')^2 = 1$.

Lemma 2.2. *Let the graph G contain a vertex v, connected to the vertices of the set $V_v = \{v_1, \ldots, v_n\}$. Split the set V_v into two disjoint subsets V_A и V_B and replace the vertex v by the 4-element path $AC_1C_2C_3B$ (where C_i are new vertices, and all the edges of this path have weight 1), and after that connect A to all the vertices of V_A, and connect B to all the vertices of V_B (the weights of these edges are equal to the weights of the corresponding edges of the the former vertex v). Denote the obtained graph by G'. Then* $\det A(G) = \det A(G')$.

Proof. Construct a bijection between 1-factors of the graph G and 1-factors of the graph G', preserving the weight of 1-factor. For this we rebuild cycles $\ldots p \to v \to q \to \ldots$, that pass through the vertex v (p and q denote two vertices of the graph G, not necessarily distinct).

If $p \in V_A$ and $q \in V_A$, we replace this cycle by the similar cycle in G', that passes through A, v_i и v_j. and append two more 2-cycles: $C_1 \to C_2 \to C_1$ и $C_3 \to B \to C_3$. The number of even cycles is increased by 2, so the contribution to the determinant does not change. The case $p \in V_B$, $q \in V_B$ is similar.

If $p \in V_A$ и $q \in V_B$, we replace this cycle by the cycle $\ldots p \to A \to C_1 \to C_2 \to C_3 \to B \to q \to \ldots$ The weight and the parity of the length of the cycle do not change, so the contribution to the determinant does not change too. $\square$

Let us mention two more statements about transformations, which preserves the determinants [6, theorems 2, 3].

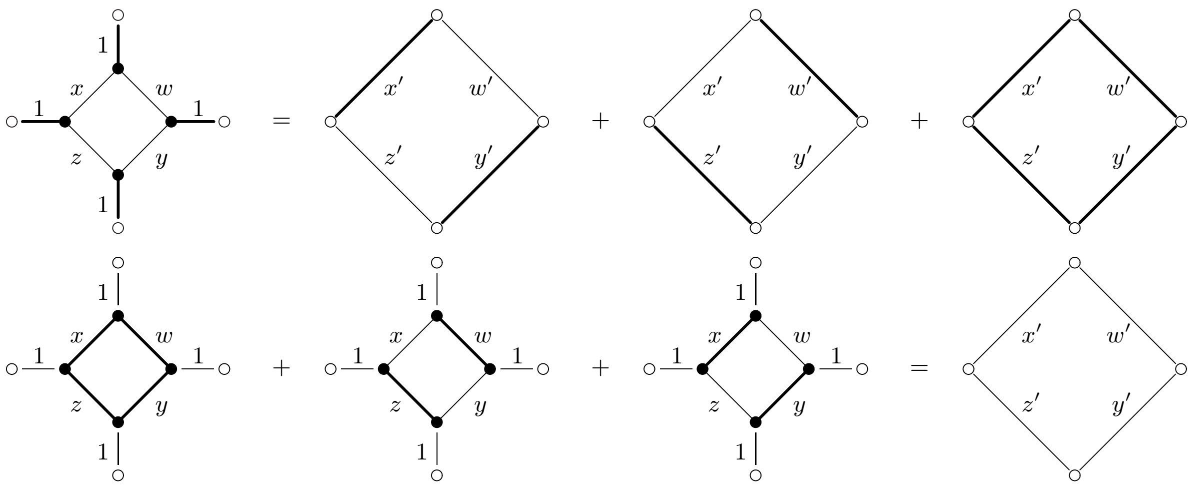


Figure 4. Remaining identities

Lemma 2.3. *Let a graph contain ∯m edge uv with the weight w. Then if we replace this edge by a path of the length 5 with edge weights 1, 1, w, 1, 1, then the determinant of the adjacency matrix of the graph does not change.*

The proof is similar to the previous one.

Lemma 2.4. *Let graph G contain a vertex v, connected to the vertices of the set V_1 and a vertex w, connected to the vertices of the set $V_1 \cup V_2$. If we remove all the edges from w to V_1 then the determinant of the adjacency matrix does not change.*

It follows from this lemma, that if we remove a hanging vertex and its edge then the determinant of the adjacency matrix should be divided by the weight of the removed edge. But this is obvious form the point of view of 1-factors.

Example 2.1. Let D_n be the aztec diamond whose edges have weights x, y, w and z as shown on the left of the picture 5. Then

$$\det A(D_n) = (xy - wz)^{n(n+1)}.$$

To prove this we repeat reasonings of Propp [5], applying lemmas 2.1, 2.2.

First, apply lemma 2.2 to each vertex of the graph and color the rhombi in a chess-coloring order (picture 6, on the left). Next, apply the urban renewal operation three times in each black rhombus, and apply it once in each white rombus. Since the iterations of the rule (1) are involutive, the obtained weights are equal to x', y', w' and z' (picture 6, on the right). As the result of all operations the determinant is multiplied by $(xy - wz)^{2n^2}$, because the total number of rhombi equals n^2. Now remove the hanging fragments of one and three edges on the boundary of the diamond. This does not affect the determinant, because the deleted edges have the weight 1. After removal of these fragments the vertices of diamond on the boundary are removed too, therefore the size of the remaining diamond is decreased by 1, and the weights arrangement is similar to original one (picture 5, on the right). This gives us the step of induction.

Example 2.2. Let us calculate the determinant of the adjacency matrix of the cylinder $C_4 \times P_{m-1}$. More precise, we want to prove that

$$\det A(C_4 \times P_{m-1}) = \begin{cases} m^2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

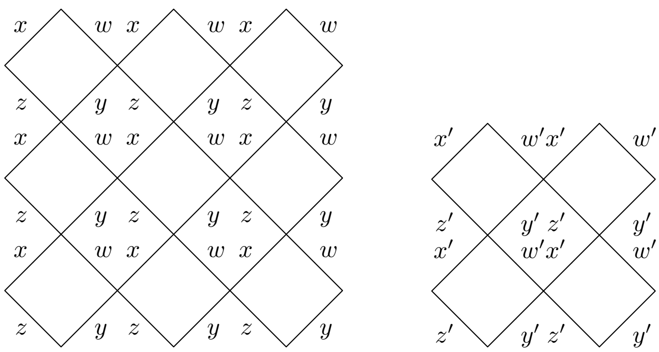


Figure 5. The diamants D_n and D_{n-1}

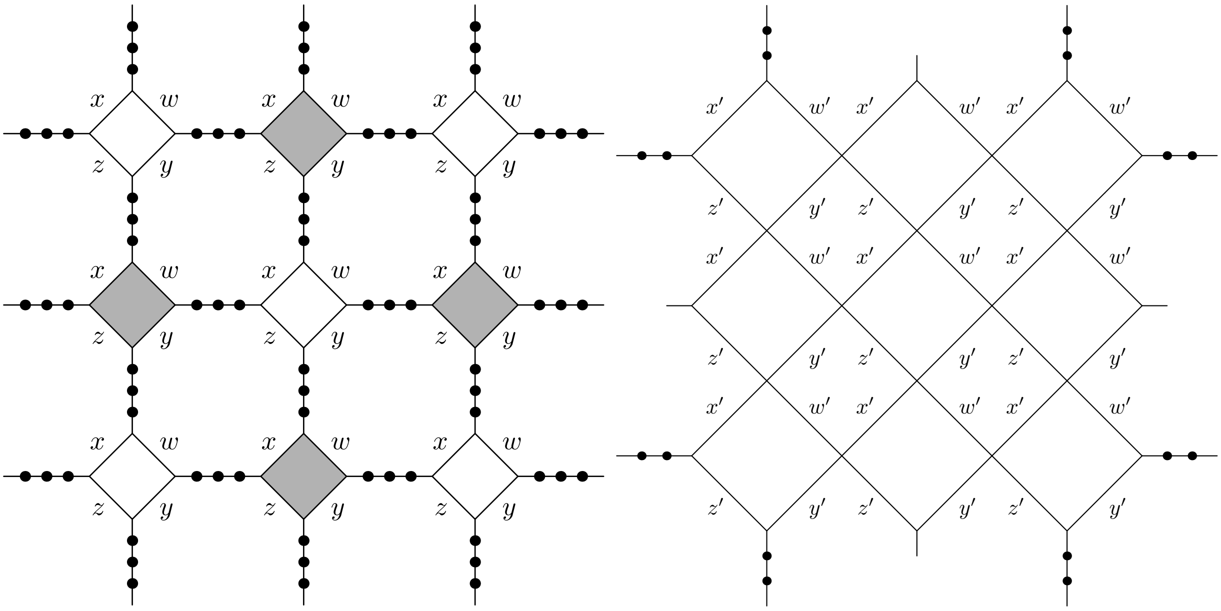


Figure 6. The transformation of the diamant

Applying lemma 2.1, we remove 4-cycles on the boundary the cylinder step by step (picture 7). To avoid zeros in denominators, we assume, that the edges of the first cycle H_0 on the boundary have weights $x_0 = a$, $z_0 = 1$, $y_0 = a$, $w_0 = 1$, and all other edges in the graph have weight 1. After one application of the lemma the cycle H_0 disappears, and we obtain on the boundary a new cycle H_1. The weights of its edges are equal to the sum of the initial weights (i.e. 1) and the new weights $\frac{a}{1-a^2}$, $\frac{1}{a^2-1}$, $\frac{a}{1-a^2}$, $\frac{1}{a^2-1}$ obtained by the formula (1). So,

$$x_1 = 1 + \frac{a}{1-a^2}, \quad z_1 = 1 + \frac{1}{a^2-1}, \quad y_1 = 1 + \frac{a}{1-a^2}, \quad w_1 = 1 + \frac{1}{a^2-1}.$$

Then one can check by induction, that after $2n$ applications of these operations (the induction step size 2 is more convenient, because for even and odd number of iterations the formulae are slightly different) the weights of edges of the boundary cycle are equal to

$$x_{2n} = y_{2n} = \frac{na^2 + a}{2na + 1}, \qquad z_{2n} = w_{2n} = \frac{1 + 2na - na^2}{2na + 1},$$

and the product of all the determinants of the removed cycles equals $\det A(H_0) \det A(H_1) \dots \det A(H_{2n-1}) = (2na + 1)^2$.

If m is even, say $m - 1 = 2n + 1$, then after $2n$ operations the remaining graph consists of the only

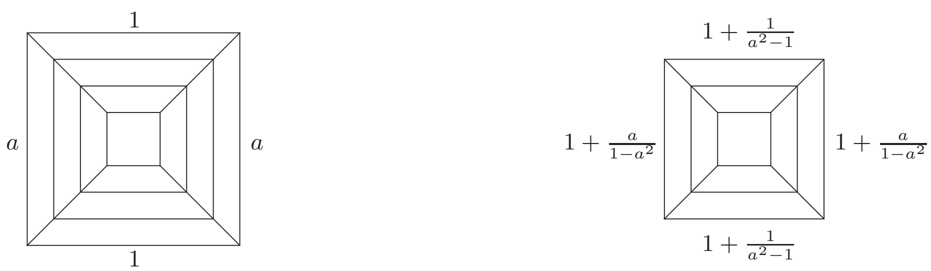


Figure 7. Removing the cycle on the border of the cylinder

4-cycle with the weights given by the formulae above. The determinant of the matrix of this cycle equals

$$(x_{2n}y_{2n} - z_{2n}w_{2n})^2 = \Big(\frac{(2n+1)a+1}{2na+1}\Big)^2 (a-b)^2,$$

and the total determinant equals

$$\det A(H_0) \det A(H_1) \dots \det A(H_{2n})(x_{2n}y_{2n} - z_{2n}w_{2n})^2 = \big((2n+1)a+1\big)^2 (a-b)^2.$$

For $a = b = 1$ this expression is equal to 0. If m is odd, by similar reasoning we get, that the determinant is equal to $\big((m-1)a+1\big)^2$, and for $a = b = 1$ this is equal to m^2.

3 *The general scheme of chip removal*

Let G be an arbitrary graph. *Chip* H is an arbitrary induced subgraph of the graph G. The vertices of the chip, which has outgoing edges (to the remaining part of the graph G) we call *outer* vertices, and the second endpoints of these edges are *contacts* (which the chip is connected to).

Let the chip H contain h vertices and k contacts, so its adjacency matrix $A(G)$ has block form:

$$A(G) = \begin{pmatrix} A(H) & K & 0 \\ K^\intercal & L & * \\ 0 & * & * \end{pmatrix}, \tag{3}$$

where K is a $h \times k$ block, that encodes connections of the chip to the remaining part of the graph, L is the $k \times k$ block (possibly zero), that encodes edges of the graph G between contacts, and the stars correspond to other possible edges outside of the chip. Multiplying matrix by $D = \begin{pmatrix} A(H)^{-1} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}$ from the left , we obtain the unity block $h \times h$ in the right-upper corner: $\begin{pmatrix} E & A(H)^{-1}K & 0 \\ K^\intercal & L & * \\ 0 & * & * \end{pmatrix}$. Then subtract the appropriate rows to make the block $K^\intercal$ to be zero, thess operations change also block L. We obtain the matrix

$$\begin{pmatrix} E & A(H)^{-1}K & 0 \\ 0 & \widetilde{L} & * \\ 0 & * & * \end{pmatrix}, \qquad \text{where} \qquad \widetilde{L} = L - K^\intercal A(H)^{-1}K. \tag{4}$$

And finally, multiplying by D^{-1} from the left, we «get back» the block $A(H)$ and obtain the matrix $\begin{pmatrix} A(H) & K & 0 \\ 0 & \widetilde{L} & * \\ 0 & * & * \end{pmatrix}$. We interpret the changes in the block L, as «repair» or «bridge installation», i. e. creating additional edges between contacts. The weights of these edges are specified in the matrix $-K^\intercal A(H)^{-1}K$. Denote the graph, obtained as a result of repair, by G'. Obviously, the determinant of the adjacency matrix does not change during these operations, and matrix $\begin{pmatrix} \widetilde{L} & * \\ * & * \end{pmatrix}$ is exactly the adjacency matrix of the repaired graph $A(G')$. Thus,

$$\det A(G) = \det A(H) \cdot \det A(G').$$

It is not hard to see, that the main obstacle for combinatorial interpretation of this algebraic transformations is a complicated form of the matrix $A(H)^{-1}$.

Example 3.1. $K_{3,3}$ removal. Let the chip H be the graph of 6 vertices «hexagon with three main diagonals», each vertex has exactly one outgoing edge, the weights of the edges on the sides of the hexagon are equal to b, and the weights of the diagonals are equals to c (picture 8).

Denote $S(b,c)$ the weighted adjacency matrix of the chip H . Surprisingly, direct calculation shows that

$$S(b,c)^{-1} = S(b',c'), \quad \text{где } b' = \frac{b}{2b^2 - bc - c^2}, \; c' = -\frac{b+c}{2b^2 - bc - c^2}.$$

So, after the chip removal the new bridges between contacts give us a new chip of the same form, where the weights of the sides are equal to $-\frac{b}{2b^2-bc-c^2}$ and weights of diagonals equal $\frac{b+c}{2b^2-bc-c^2}$.

Example 3.2. Let us calculate the determinant of the cylinder $\det A(C_6 \times P_{m-1})$. We will show that

$$\det A(C_6 \times P_{m-1}) = \begin{cases} (-1)^{m-1}m^2 & \text{if } m \text{ is not divisible by 3,} \\ 0 & \text{if } m \text{ is divisible by 3.} \end{cases}$$

To prove this we consequently apply the operation from the previous example, where in each step we choose a chip as a 6-vertices graph on the boundary of the cylinder.

Let $C = S(1,0)$ be the adjacency matrix of the 6-cycle (notations from the previous example) and let A_k be the adjacency matrix of the 6-vertices graph, located on the border of the cylinder after application of k steps, in particular, $A_0 = C$. By virtue of (4)

$$A_{k+1} = C - A_k^{-1}.$$

Solving this recurrence relation, we find, that $A_k = U_k(C/2)U_{k-1}(C/2)^{-1}$, where U_k is the Chebyshev polynomals of the second kind. Thus,

$$\det A(C_6 \times P_{m-1}) = \det A_0 \cdot \det A_1 \cdot \ldots \cdot \det A_{m-1} = \det U_{m-1}(C/2).$$

The determinant of the matrix is the product of the eigenvalues, eigenvalues of the matrix C are equal to $\cos\frac{2j\pi}{6}$, $j = 1, \ldots, 6$. By the definition of the Chebyshev polynomial, $U_{m-1}(\cos\theta) = \frac{\sin m\theta}{\sin\theta}$. So,

$$\det A(C_6 \times P_{m-1}) = \prod_{j=1}^{6} \frac{\sin\frac{jm\pi}{3}}{\sin\frac{j\pi}{3}}.$$

(Incorrect fractions for $m = 3, 6$ should be understood by continuity.) For m, divisible by 3, many sinuses are equal to 0; for m, not divisible by 3, all, except incorrect fractions, are reducible, and we obtain the answer.

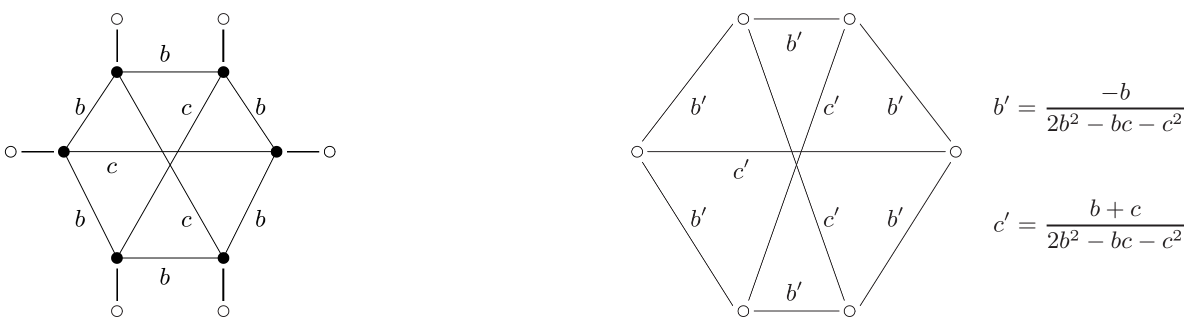


Figure 8. $K_{3,3}$ and the result of its removal

Example 3.3. The graph $K_{4,4}$ with the weights of four different types (picture 9) can be also removed with reasonable recurrence. Let us denote the weighted adjacency matrix of this type as $S(a,b,c,d)$. The straightforward calculation shows that

$$S(a,b,c,d)^{-1} = S(a',b',c',d),$$

where $a' = \frac{1}{\Delta}(a^3 - 2adb - c^2a + d^2c + b^2c)$, $b' = \frac{1}{\Delta}(d^2b - da^2 + 2cba - c^2d - b^3)$, $c' = \frac{1}{\Delta}(d^2a - ca^2 + b^2a + c^3 - 2cdb)$, $d' = -\frac{1}{\Delta}(db^2 - a^2b + 2dca - d^3 - c^2b)$, $\Delta = \det S(a,b,c,d) = (a+b+c+d)(a-b+c-d)(b^2 - 2db - 2ca + d^2 + a^2 + c^2)$. So, after $K_{4,4}$ removal, one should add bridges to the contacts in such a way, that they form the graph $K_{4,4}$ with weights $-a'$, $-b'$, $-c'$, $-d'$.

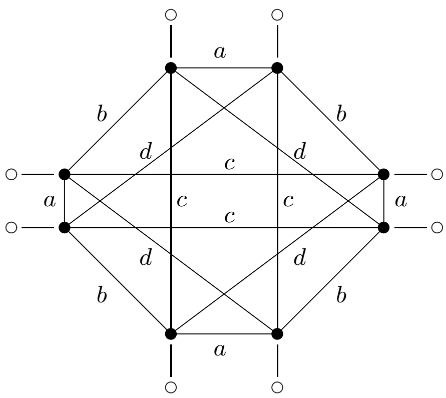


Figure 9. $K_{4,4}$ removal

4 *Rectangle removal*

Definition. Denote $H_{n,m}$ a rectangle-shaped chip which is an induced subgraph of the graph G of the form of $P_n \times P_m$. We assume for convenience that the chip is constructed by the grid lines. We assume also that the chip is connected to the remaining part of the graph by $2n$ outer edges — n horizontal segments, going to the left, and n horizontal segments, going to the right (picture 10).

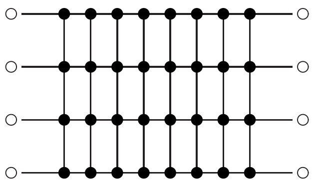
For example the removal of $H_{2,2}$ chip is the same as Urban Renewal operation.

Let A be the adjacency matrix of the path P_m. Then the adjacency matrix of the rectangular chip H_{nm} has the form of the $n \times n$ block matrix

$$A(H_{n,m}) = \begin{pmatrix} A & E & 0 & \dots & 0 \\ E & A & E & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & E & A \end{pmatrix}, \qquad \text{where} \quad A = A(P_m).$$

Lemma 4.1. *Let $U_n(y)$ be the Chebyshev polynomial of the second kind. Assume that the number $2a \in \mathbb{R}$ is not its root and let $B(a)$ be $n \times n$ matrix of the following type* $B(a) = \begin{pmatrix} a & 1 & 0 & \dots & 0 & 0 \\ 1 & a & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a & 1 \\ 0 & 0 & \dots & 0 & 1 & a \end{pmatrix}$. *Then*

1) $\det B(a) = U_n(a/2)$. *In particular, matrix $B(a)$ is invertible.*



$$\begin{array}{ccccc|ccc}
A & E & \dots & 0 & 0 & E & 0 & 0\\
E & A & \dots & E & 0 & 0 & 0 & 0\\
\dots & & \dots & & \dots & \dots & \dots & \dots\\
0 & E & \dots & A & E & 0 & 0 & 0\\
0 & 0 & \dots & E & A & 0 & E & 0\\
\hline
E & 0 & \dots & 0 & 0 & L_{11} & L_{12} & *\\
0 & 0 & \dots & 0 & E & L_{21} & L_{22} & *\\
0 & 0 & \dots & 0 & 0 & * & * & *
\end{array}$$

Figure 10. The rectangular chip 4×8 and its adjacency matrix as a block in the matrix $A(G)$ and the result of chip removal

2) Let us denote $B(a)^{-1} = (x_{ij})$. *Then* $x_{11} = x_{nn} = \frac{U_{n-1}(a/2)}{U_n(a/2)}$, $x_{1n} = x_{n1} = \frac{(-1)^{n+1}}{U_n(a/2)}$.

3) Let $B = \frac{1}{2}B(0)$, E *be the identity* $n \times n$ *matrix and* $\ell = 2km - 2$, *when* $m = n + 1$. *Then*

$$U_{\ell-1}(B)(U_\ell(B))^{-1} = 2B, \qquad -(U_\ell(B))^{-1} = E. \tag{5}$$

Proof. Statement 1) is a well known fact. The formula of the statement 2) can be obtained instantly from the Kramer's formulae the elements of the inverse matrix.

We check the identities 3) in the basis consisting of the eigenvectors of the matrix B. The eigenvalues of the matrix $B(0)$ are $2\cos\frac{j\pi}{m}$, $j = 1, \dots, m-1$. By the definition of the Chebyshev polynomials $U_\ell(\cos\theta) = \frac{\sin(\ell+1)\theta}{\sin\theta}$. Then the identities (5) follow from the trigonometric identities

$$\sin\frac{(2km-2)\pi j}{m} = 2\cos\frac{\pi j}{m} \cdot \sin\frac{(2km-1)\pi j}{m}, \qquad \sin\frac{(2km-1)\pi j}{m} = -\sin\frac{\pi j}{m}. \qquad \square$$

In particular, we have

$$\det H_{n,m} = \det U_n\Big(\frac{1}{2}A\Big). \tag{6}$$

Example 4.1. Let us consider the $(2km-2) \times (m-1)$ chip removal.

So, let the graph G contain the chip $H = H_{(2km-2),(m-1)}$, connected to $(m-1)$ «contacts». Denote for brevity $\ell = 2km-2$. The matrix $A(G)$ in the block form is shown on the picture 10. (We restrict ourselves by 8×4 example i. e. here $m = 5$, $k = 2$.) The stars denote blocks of the matrix $A(G)$, corresponding to the edges between the vertices $G \setminus H$. All blocks except the blocks in the right column and bottom row, have the dimensions $(m-1) \times (m-1)$. The adjacency matrix $A(H)$ is cut by the auxiliary lines in the left-bottom corner.

Due to (6) and the second identity (5) the matrix $A(H)$ is invertible and

$$\det A(H_{(2km-2),(m-1)}) = (-1)^{m-1}.$$

Let $A(H)^{-1} = (X_{ij})$ be $\ell \times \ell$ block matrix with the blocks of size $(m-1) \times (m-1)$. Then by lemma 4.1

$$\begin{aligned} X_{11} &= X_{\ell\ell} = U_{\ell-1}(A/2)U_\ell(A/2)^{-1} = A, \\ X_{1\ell} &= X_{\ell 1} = -U_\ell(A/2)^{-1} = E. \end{aligned} \tag{7}$$

Now, following the general chip removal scheme, we only need to understand what changes we have in the block $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$, that describes the new connections between contacts. Due to (4) the elements of block L are changed by the formula

$$\widetilde{L} = \begin{pmatrix} L_{11} - X_{11} & L_{12} - X_{1\ell} \\ L_{21} - X_{\ell 1} & L_{22} - X_{\ell\ell} \end{pmatrix}. \tag{8}$$

So, due to equations (7) the repair after the chip removal is surprisingly easy: we need to install a few bridges between contacts (picture 10), the weights of the new bridges are equal to -1 (if a new bridge duplicates already existing edge, then its weight is just added to the weight of the edge).

Example 4.2. In examples 2.2, 3.2 we have already calculated the determinants of the cylinders. Now we calculate the determinant of the adjacency matrix of an arbitrary cylinder $C_n \times P_{m-1}$, to be more precise, we check, that

$$\det A(C_n \times P_{m-1}) = \begin{cases} m & \text{if } n \text{ is odd and } \mathrm{GCD}(m,n) = 1, \\ (-1)^{m-1} m^2 & \text{if } n \text{ is even and } \mathrm{GCD}(m,n/2) = 1, \\ 0 & \text{in other cases.} \end{cases}$$

This result is known, see [2], where this calculation is done in totally algebraic way.

Let us remove from the cylinder a $(n-1) \times (m-1)$ rectangular chip. After the removal the only one column of vertices remains in the graph, but after repair we will have also some new edges. The adjacency matrix of the cylinder $A(C_n \times P_{m-1})$ can be written in the block form

$$A(C_n \times P_{m-1}) = \left(\begin{array}{ccccc|c} A & E & 0 & \dots & 0 & E \\ E & A & E & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & E & A & E \\ \hline E & 0 & \dots & 0 & E & L \end{array}\right).$$

The upper-left part of this matrix is the adjacency matrix of the chip $H_{n-1,m-1}$, and the matrix L in the bottom-right corner is the adjacency matrix of the remaining column, therefore $L = A$.

Denote for brevity $f_n(x) = U_n(x/2)$, $g_n(x) = T_n(x/2)$, where T_n, U_n are the Chebyshev polynomials of the first and second kind. Let $A(H_{n-1,m-1})^{-1} = (X_{ij})$, then similarly to the formulae (7) (in our case $\ell = n-1$ and we use the second statement from lemma 4.1 only) we have:

$$X_{1,1} = X_{n-1,n-1} = f_{n-2}(A) \cdot f_{n-1}^{-1}(A), \quad X_{1,n-1} = X_{n-1,1} = (-1)^n f_{n-1}^{-1}(A).$$

After the chip removal and the repair we have

$$A(G') = \widetilde{L} = A - X_{1,1} - X_{1,n-1} - X_{n-1,1} - X_{1,n-1}.$$

And now we can calculate the determinant:

$$
\begin{aligned}
\det A(C_n \times P_{m-1}) &= \det H_{n-1,m-1} \cdot \det(A - X_{1,1} - X_{1,n-1} - X_{n-1,1} - X_{1,n-1}) = \\
&= \det(f_{n-1}(A)) \cdot \det\bigl(A - 2 \cdot f_{n-2}(A) \cdot f_{n-1}^{-1}(A) - 2 \cdot (-1)^n \cdot f_{n-1}^{-1}(A)\bigr) = \\
&= \det\bigl(A \cdot f_{n-1}(A) - 2 \cdot f_{n-2}(A) - 2 \cdot (-1)^n \cdot E\bigr) = \\
&= \det\bigl(f_n(A) - f_{n-2}(A) - 2 \cdot (-1)^n \cdot E\bigr) = \\
&= \det\bigl(2g_n(A) - 2 \cdot (-1)^n \cdot E\bigr) = 2^{m-1} \det\bigl(g_{n-1}(A) - (-1)^n \cdot E\bigr). \qquad (9)
\end{aligned}
$$

Here we use the recurrence relation for the Chebyshev polynomials and the formula $U_n - U_{n-2} = 2T_n$.

The eigenvalues of the matrix $A = A(P_{m-1})$ are the numbers $\{2\cos\frac{j\pi}{m}, j = 1, \ldots, m-1\}$. Since $g_n(2\cos(\theta)) = \cos(n \cdot \theta)$, and since the determinant is equal to the product of the eigenvalues, we obtain that

$$\det\bigl(g_{n-1}(A) - (-1)^n \cdot E\bigr) = \prod_{i=1}^{m-1}\left(\cos\Bigl(\frac{ni\pi}{m}\Bigr) - (-1)^n\right).$$

If n — even, $n' = \frac{n}{2}$ и $\mathrm{GCD}(\frac{n}{2}, m) = 1$, we obtain

$$\det A(C_n \times P_{m-1}) = 2^{m-1}\prod_{i=1}^{m-1}\left(\cos\Bigl(\frac{ni\pi}{m}\Bigr) - (-1)^n\right) = (-1)^{m-1}4^{m-1}\prod_{i=1}^{m-1}\sin^2\Bigl(\frac{n'i\pi}{m}\Bigr) = (-1)^{m-1}m^2.$$

If n — odd and $\mathrm{GCD}(n, m) = 1$, then

$$\det A(C_n \times P_{m-1}) = 2^{m-1}\prod_{i=1}^{m-1}\left(\cos\Bigl(\frac{ni\pi}{m}\Bigr) + 1\right) = 4^{m-1}\prod_{i=1}^{m-1}\sin^2\Bigl(\frac{n(2m-i)\pi}{2m}\Bigr) = 4^{m-1}\prod_{i=1}^{2m-1}\sin\Bigl(\frac{ni\pi}{2m}\Bigr) = m.$$

In other cases the product is equal to zero.

Example 4.3. Let us calculate the adjacency matrix of the torus $C_n \times C_m$, to be more precise, we check that

$$\det A(C_n \times C_m) = \begin{cases} 4^{\mathrm{GCD}(m,n)} & \text{if } m \text{ and } n \text{ are odd,} \\ 0 & \text{in other cases.} \end{cases}$$

This result is also known [2].

Similarly to the previous example, we remove from the torus a cylindrical chip $P_{n-1} \times C_m$. As in the formula (9),

$$\det A(C_n \times P_{m-1}) = 2^m \det\bigl(g_{n-1}(A) - (-1)^n \cdot E\bigr),$$

where A is the adjacency matrix of the cycle of the length m. We also know that the set of eigenvalues of the matrix A is the set $\{2\cos\frac{2j\pi}{m},\ j = 1, \ldots, m\}$. Hence,

$$\det A(C_n \times P_{m-1}) = 2^m\prod_{i=1}^{m}\left(\cos\Bigl(\frac{2ni\pi}{m}\Bigr) - (-1)^n\right).$$

If n or m is even, then the product is equal to zero. Otherwise, let $d = \mathrm{GCD}(n, m)$, $n' = \frac{n}{d}$ and $m' = \frac{m}{d}$.

Then the last product is equal to

$$2^m \prod_{i=1}^{m} \left(\cos\left(\frac{2ni\pi}{m}\right) + 1 \right) = 4^m \prod_{i=1}^{m} \cos^2\left(\frac{ni\pi}{m}\right) = 4^m \left(\prod_{i=1}^{m'd} \cos\left(\frac{n'i\pi}{m'}\right) \right)^2 = 4^m \left(\frac{1}{4^{m'-1}} \right)^d = 4^d.$$

5 *The application to pfaffians*

Consider a modification of the lemma 2.4 first. Let the graph G have a pfaffian orientation, and u, v, w be its three vertices, such that the graph contains edges uw and vw. We say that the vertex w is *significant* (with respect to the pair u, v), if the ordered pairs (u, w), (v, w) are both in the right order or both in the wrong order accordingly the pfaffian orientation, otherwise we call the vertex w *insignificant*.

Lemma 5.1. *Assume that a graph G have a pfaffian orientation. Let vertex v is adjacent to the vertices of the set V_1 and vertex w is adjacent to the vertices of the set $V_1 \cup V_2$, and the weights of all the edges, which connect v and w to insignificant vertices, are equal to 1. Let G' be the graph, obtained from G by the following operations: remove all the edges, which connect w with insignificant vertcies from the set V_1, and increase by 1 weights of all the edges, which connect w with the significant vertices from the set V_1. Then $M(G) = M(G')$.*

Proof. Write the antisymmetric adjacency matrix of the graph G, according to the pfaffian orientation. Let the first row and the first column of this matrix correspond to v, and the second row and the second column correspond to v. By subtracting the second row from the first row and the second column from the first column, we obtain the desired result. □

Now let us look at the chip removal, if we try to use that operation for calculating the number of matchings.

Example 5.1. The 2×3 chip removal. Let G be a graph on the square grid. Fix the pfaffian orientation of the graph G. Let the graph G contains chip $H_{2,3}$, depicted on the picture 11. Assume also, that the graph G does not contain the edges, which connect the contacts.

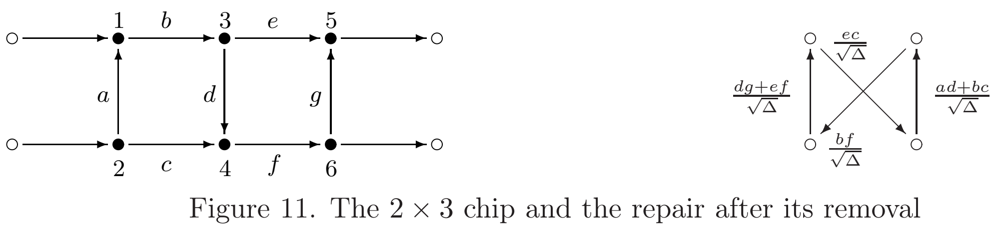


Figure 11. The 2×3 chip and the repair after its removal

We denote by $\widetilde{A}(G)$ the antisymmetric adjacency matrix of the graph G, which corresponds to pfaffian orientation. Then $\Delta = \det \widetilde{A}(H_{2,3})$ is the square of the number of weighted matchings in the chip $H_{2,3}$. So, $\Delta = (agd + gcb + afe)^2$. Apply the operation of chip removal to calculate determinant of skew matrix $\det \widetilde{A}(G)$. The distinction with symmetric case is negligible due to the antisymmetry. The straightforward calculation shows, that after the chip removal, one should append the bridges, shown on the left of the picture 11. Denote the graph obtained after the repair by G', we obtain that

$$\det \widetilde{A}(G) = \det \widetilde{A}(H_{2,3}) \cdot \det \widetilde{A}(G').$$

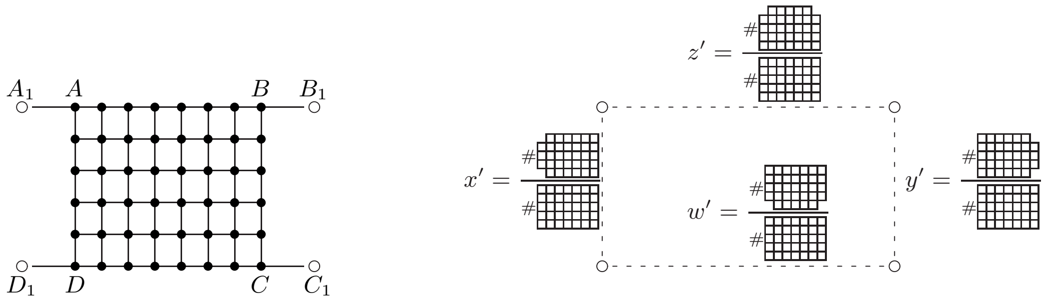


Figure 12. $2n \times 2m$ four-contact chip removal when calculating the number of matchings

Or with root extraction,

$$\operatorname{Pf}\widetilde{A}(G) = \operatorname{Pf}\widetilde{A}(H_{2,3}) \cdot \operatorname{Pf}\widetilde{A}(G').$$

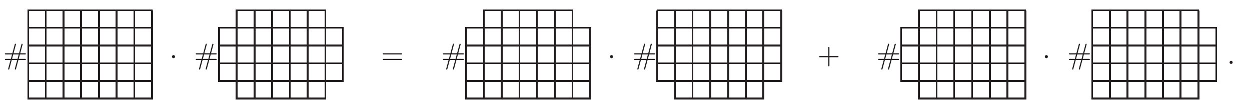
In this equalities one should understand, that the value $\det A(G')$ doesn't equal to the square of the number of the weightd matchings of the graph G', because the new (diagonal) edges violate the pfaffian orientation, and so not all the matchings in the graph G' have the same sign.

We can avoid these issues if we consider the chips, with 4 contacts only, such that in chess coloring of the vertices one diagonal pair is colored in white and the other one is colored in black.

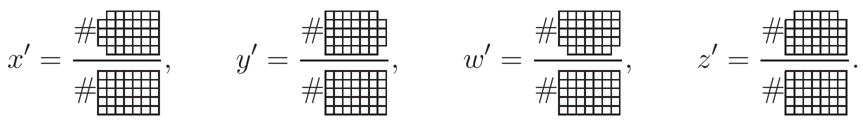
We will denote the number of matchings of the graph not only by the usual designation $M(G)$, but also by the symbol # and schematic depiction of the graph.

The following lemma is key for Kuo's graphical condensation.

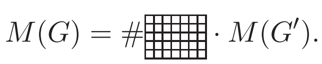
Lemma 5.2 ([4, theorem 2.1])**.** *The number of matchings of the $2n \times 2m$ chip and the numbers of matchings of the figures, obtained by removing two or four corner vertices, satisfy the identity*



Theorem 5.3. *Let the rectangular $2n \times 2m$ chip in the graph G have 4 contacts only and these contacts be connected to the corner vertices of the chip. Denote by G' the graph, obtained by the chip removal and the repair (see picture 12), with the weights of new edges*

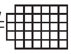

(10)

Then


(11)

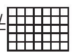
Proof. Denote the chip by H. Let us construct a bijection between the matching of the graphs G and G' as in the proof of the lemma 2.1. Denote the corner vertices of the chip by A, B, C, D, and the corresponding contacts by A_1, B_1, C_1, D_1 (picture 12). Since both sizes of the chip are even the vertices A and C have the same color in chess coloring (say white), and the vertices B and D are of the opposite color (black). The chip contains equal number of the white and black vertices. Therefore any matching of the graph G contains even number of edges AA_1, BB_1, CC_1, DD_1. Moreover, if the matching contains exactly two of edges listed above, then their endpoints are the endpoints of a side of the rectangle $ABCD$.

Let us consider the cases.

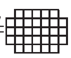
1) Consider all the matchings of the graph G, that contain both edges AA_1 and DD_1 and does not contain both edges BB_1, CC_1. Fix some placement of edges of matching outside the chip, i. e. in fact we fix the matching of the graph $G' \setminus \{A_1, D_1\}$ — denote by μ' this matching, and by $w(\mu')$ its weight. If we combine the configuration, consisting of the edges AA_1, DD_1 and all the edges of the matching μ', with an arbitrary matching of the graph $H \setminus \{A, D\}$, we obtain a matching of the whole graph G. Denote by μ the set of matchings of the graph G, obtained by this way, the weighted sum of these matchings is equal to #  $\cdot\, w(\mu')$.

The set μ of these matchings of the graph G we put into correspondence to one (!) matching from the graph G', namely, the matching, that contains all the edges from the set μ' (with the same weights) and the new edge A_1D_1 of weight x' (as in (10)). Obviously, the weight of this matching is equal to $x'w(\mu')$ and after multiplying by the term on the right hand side of (11) we obtain the equality of weights.

The similar construction works for matchings that contain other pairs of edges.

2) Consider all the matchings of the graph G, which does not contain any of edges AA_1, BB_1, CC_1, DD_1. Fix some placement of edges of matching outside the chip — i. e. we fix the matching of the graph G', denote by μ' this matching and by as $w(\mu')$ its weight. If we combine the configuration, consisting of all the edges from the set μ', with an arbitrary matching of the graph H we obtain a matching of the whole graph G. Denote by μ the set of matchings of the graph G, obtained by this way. The weighted sum of these matchings is equal to #  $\cdot\, w(\mu')$. The set of these matchings μ of the graph G we put into correspondence to one matching from the graph G', namely, the matching that contains all the edges from the set μ' (with the same weights). Obviously, the corresponding objects give the same contribution to the r.h.s. and the l.h.s of the formula (11).

3) Finally, consider matchings of the graph G, which contain all the four edges AA_1, BB_1, CC_1, DD_1. Fix a placement of edges of matchings outside the chip,i. e. the matching of the graph $G' \setminus \{A_1, B_1, C_1, D_1\}$ — denote by μ' this matching, and by $w(\mu')$ its weight. If we combine the configuration that consists of the edges AA_1, BB_1, CC_1, DD_1 and all the edges from the set μ' with an arbitrary matching of the graph $H \setminus \{A, B, C, D\}$ we obtain a matching of the whole graph G.

Denote by μ the set of matchings of the graph G, obtained by this way, the weighted sum of these matching is equal to #  $\cdot\, w(\mu')$. Put the set of the matchings μ in the graph G into correspondence to the pair of the matchings of the graph G', namely, the matching, that consists of the edges from the set μ' (with the same weight) and the edges A_1D_1, B_1C_1 (with the weights x', y'), and the matching, that consists of the edges from the set μ' and the edges A_1B_1, C_1D_1 (with the weights z', w'). The sum of weights of these matchings is equal to $(x'y' + w'z')w(\mu')$

Once again, the corresponding objects give the same contribution to the r.h.s and the l.h.s of (11) due to lemma 5.2. $\square$

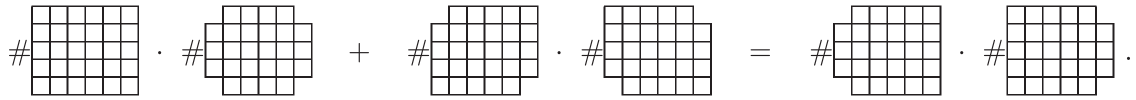
Remark. The statement similar to lemma 5.2 is proved in [4] for arbitrary planar bipartite graphs. It allows to generalize theorem 5.3 for the four-contacts chip of any form, for which the «diagonally-opposite» contacts have the same color in chess-coloring.

If the opposite contacts of a chip have different colors, for example A_1 and B_1 are white, C_1 and D_1 are black, the following lemma similar to lemma 5.2 holds (we restrict ourselves with the case of rectangular chips).

Lemma 5.4 ([4, theorem 2.3]). *The number of matchings of the* $2n \times (2m + 1)$ *chip and the numbers of*

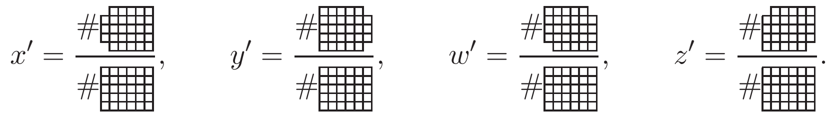
Figure 13. The four contacts $2n \times (2m+1)$ chip removal, when calculating the number of matchings

matchings of the figures, obtained by removing its two or four corner vertices, satisfy the identity

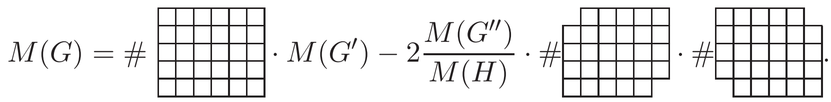


If we try to prove an analog of the theorem 5.3 in this case, we should keep in mind the issue, which we observe in the example 5.1. It is not difficult to fix the reasoning in the point 3) of the proof of the theorem 5.3, and obtain the following statement.

Theorem 5.5. *Let a rectangular $2n \times (2m+1)$ chip H in the graph G have 4 contacts only and these contacts be connected to the corner vertices of the chip. Denote by G' the graph, obtained from the graph G by removing this chip and by repairing the graph as shown on the right of the picture 13, with the weights of the edges*



Moreover, denote by G'' the graph, obtained from the graph G by removing the chip H and four vertices A_1, B_1, C_1, D_1. Then



Let us give one more example of the application of the chip removal technique.

Example 5.2. The next statement is well-known from times of the work of Cuicu [3].

Theorem 5.6. *If $N+1$ is divisible by $n+1$ and $M+1$ is divisible by $m+1$, then $\#(P_M \times P_N)$ is divisible by $\#(P_m \times P_n)$.*

Proof. It is sufficient to consider a case $M = m$. Let $N+1 = (k+1)(n+1)$. Remove from the $m \times (kn+k+n)$ rectangle $k+1$ copies of $m \times n$ chip simultaneously (see picture 14 when $k=2$). The resulting graph G' consists of k copies of the graph P_m and new bridges inside each copy P_m, and new bridges between neighboring copies.

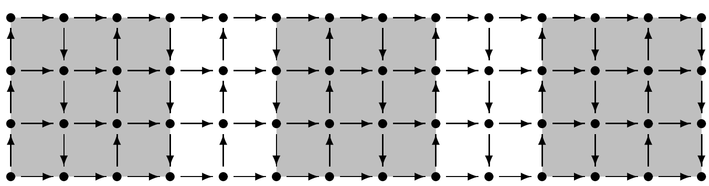


Figure 14. Remove a lot of chips

Consider antisymmetric adjacency matrices, which correspond to pfaffian orientation: $A(P_m)$, $A(H_{m,n})$, $A(P_m \times P_N)$ and $A(G')$; let $X = A(H_{m,n})^{-1}$. If we calculate $\det A(H_{m,n})$ in the block form, we obtain similarly to calculations in the example 4.2, that $\det A(H_{m,n}) = \det F(A)$, where F is some polynomial with integer coefficients (actually it is $U_n(x/2)$ again up to a sign). Therefore, the matrix $F(A)$ is integer. Let A_k be the block-diagonal matrix, which contains k blocks $F(A)$ on the diagonal. Obviously, $\det A_k = \det A(H_{m,n})^k$.

The matrix $A(G')$ is obtained by «the repair» from the block-diagonal matrix, which contains k blocks $\pm A(P_m)$ on the diagonal. During the repair we add «correction» blocks to blocks on the main diagonal and two neighbouring diagonal rows. Similar to (8) these «correction» blocks are determined by the matrices $\pm X_{1,1}$, $\pm X_{1,n}$, $\pm X_{n,1}$, $\pm X_{n,n}$. By the Kramer's formula for elements of the inverse matrix, we obtain that the matrices $F(A)X_{1,1}$, $F(A)X_{1,n}$, $F(A)X_{n,1}$, $F(A)X_{n,n}$ are integer. So, the matrix $A_kA(G')$ is integer.

Thus, as a result of removing the chips we obtain the identity

$$\det A(P_m \times P_N) = \big(\det A(H_{m,n})\big)^{k+1} \det A(G') = \det A(H_{m,n}) \det(A_k A(G')),$$

which shows, that $\det A(P_m \times P_N)$ is divisible by $\det A(H_{m,n})$. Since the matrices are antisymmetric, the proof works for the pfaffians, too.

References

[1] *Aksenov V., Kokhas K.* Domino tilings and determinants // Journal of Mathematical Sciences, vol. 200, no. 6, 647–653. (2014)

[2] *Bibak Kh., Tauraso R.* Determinants of grids, tori, cylinders and möbius ladders // `http://arxiv.org/abs/1212.4816v1`

[3] *Ciucu M.* Enumeration of perfect matchings in graphs with reflective symmetry. // J. Combin. Theory. Ser. A. 1997. Vol. 77. No. 1. P. 67–97.

[4] *Kuo E.* Application of graphical condensation for enumerating matchings // Teoret. Comp. Sci. Vol. 319. 2004. P. 29–57. http://www.arxiv.org/math.CO/0304090.

[5] *Propp J.* Generalized domino-shuffling // Theoret. Comput. Sci. 303, no. 2-3, 267–301, Tilings of the plane (2003). `http://arxiv.org/abs/math/0111034`

[6] *Rara H.M.* Reduction procedures for calculating the determinant of the adjacency matrix of some graphs and the singularity of square planar grids // Discrete Math. V. 151. 1996. P. 213–219.

Aksenov Vitaly Evgenievich, University ITMO, the masters student

Kokhas Konstantin Petrovich, University ITMO, associate professor